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CLOSED FORM SOLUTION OF STRESS AND STRAIN CONCENTRATION FACTORS OF AXIALLY SYMMETRIC BODIES WITH SMALL SCALE PLASTICITY BY MEANS OF THE CONSERVATIVE INTEGRAL METHOD

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Abstract—A conservative integral of axially symmetric bodies is established and applied to the determination of stress and strain concentration factors. Under the condition of small scale plasticity, a closed form solution of the above two factors can be obtained with given elastic stress concentration factors for arbitrary hardening materials. It can be verified by calculations that the results given by this method agree with the average values given by Neuber's method and other empirical formulas. The computations in this method are very time-saving and can be carried out even by calculators.

1. INTRODUCTION

It is known that the elasto-plastic stress and strain concentration factors can be determined by Neuber's formula [see e.g. Neuber (1961)]. However, this formula can only be verified under the condition of anti-plane shear for power hardening materials. There are three empirical formulas [see e.g. Seeger *et al.* (1977)] used to find the above two factors. However, the discrepancies among the results given by the above empirical formulas are unacceptable. Numerical methods can be applied to solve this problem, but they are time-consuming in the elasto-plastic stress analysis. To improve the efficiency of computational methods, a closed form solution for elasto-plastic stress and strain concentration factors is given. It employs conservative integral methods and applies to axially symmetric problems under the condition of small scale plasticity with given linear elastic stress concentration factors.

2. GOVERNING EQUATIONS

For axially symmetric problems under the condition of infinitesimal deformation, the equations of deformation geometry and equations of equilibrium are as follows:

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \varepsilon_{\theta\theta} = \frac{u_r}{r}, \quad \varepsilon_{zz} = \frac{\partial u_z}{\partial z}, \quad \varepsilon_{rz} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) \tag{1}$$

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) + F_r = 0, \quad \frac{\partial \sigma_{rz}}{\partial r} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{1}{r}\sigma_{rz} + F_z = 0, \tag{2}$$

where u_i, ε_{ij}, σ_{ij} and F_i are components of displacement, strain, stress and body force respectively, as shown in Fig. 1.

Under the assumption of proportional loading, deformation theory can be used to establish the constitutive equations. Then, we have

$$\sigma_{rr} = \frac{\partial W}{\partial \varepsilon_{rr}}, \quad \sigma_{\theta\theta} = \frac{\partial W}{\partial \varepsilon_{\theta\theta}}, \quad \sigma_{zz} = \frac{\partial W}{\partial \varepsilon_{zz}}, \quad \sigma_{rz} = \frac{\partial W}{\partial \varepsilon_{rz}}, \tag{3}$$

where W is strain energy density.

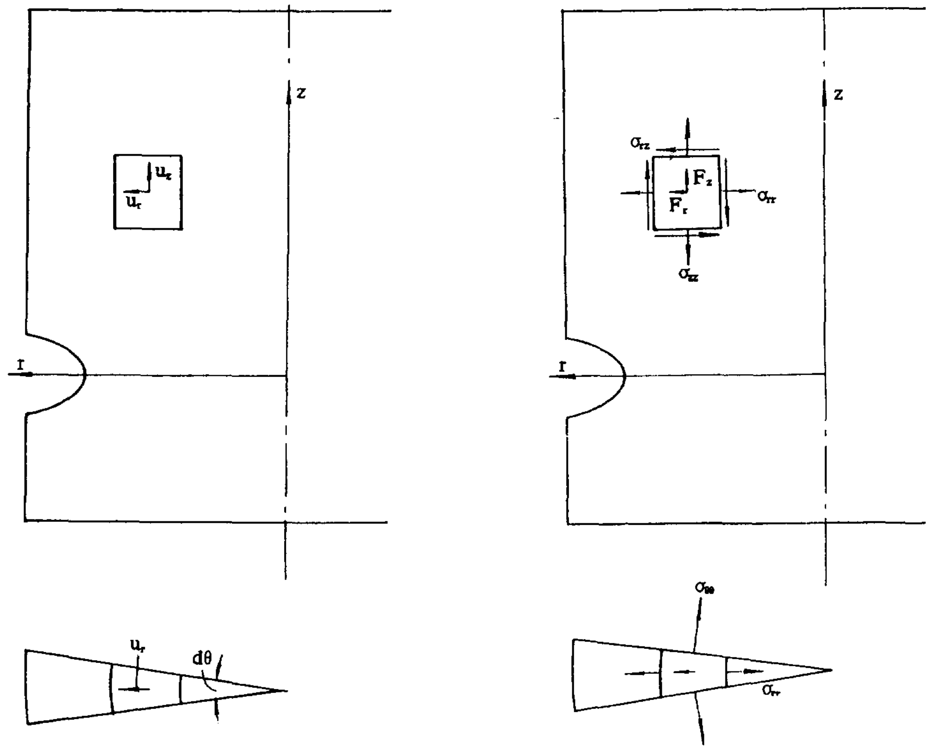

Fig. 1. Displacement, stress and body force components in axially symmetric problems.

3. CONSERVATIVE INTEGRAL

To obtain the closed form solution of elasto-plastic stress and strain concentration factors of axially symmetric problems, a corresponding conservative integral is introduced.

Figure 2 shows a circular cylinder with a circumferential notch, subjected to axial tension, as an example of axially symmetric problems.

A thin wedge is drawn from this cylinder by two planes passing through the axis of symmetry. The dihedral angle between these two planes is $d\theta$. This wedge is shown in Fig. 3.

To establish the conservative integral, a closed curve ABCDEFGA including notch bottom is drawn and denoted by Γ^*. The area surrounded by Γ^* is A_Γ. A_Γ contains three zones as follows:

(i) plastic zone A_p with boundary AHGA;
(ii) elastic zone A_{ei} influenced by a plastic zone with boundary ABIFGHA;
(iii) elastic zone A_{eu} uninfluenced by a plastic zone with boundary BCDEFIB, as shown in Fig. 3.

Furthermore, the closed curve Γ^* can be divided into two segments:

(i) boundary curve FGAB of plastic zone A_p and influenced elastic zone A_{ei} along the bottom of the notch, denoted by Γ_ρ;
(ii) boundary curve BCDEF along Γ^*, denoted by Γ, as also shown Fig. 3; then, $\Gamma^* = \Gamma + \Gamma_\rho$.

For the area A_Γ surrounded by Γ^*, there is no boundary with prescribed displacement components. If u_r, u_z and ε_{rr}, $\varepsilon_{\theta\theta}$, ε_{zz}, ε_{rz} are the true displacement and strain components respectively, then $\partial u_r/\partial x_1$, $\partial u_z/\partial x_1$ and $\partial \varepsilon_{rr}/\partial x_1$, $\partial \varepsilon_{\theta\theta}/\partial x_1$, $\partial \varepsilon_{zz}/\partial x_1$, $\partial \varepsilon_{rz}/\partial x_1$ will be kinematically admissible ones, i.e. eqn (1) will be valid and we have

$$\frac{\partial \varepsilon_{rr}}{\partial x_1} = \frac{\partial}{\partial r}\left(\frac{\partial u_r}{\partial x_1}\right), \quad \frac{\partial \varepsilon_{\theta\theta}}{\partial x_1} = \frac{1}{r}\left(\frac{\partial u_r}{\partial x_1}\right), \quad \frac{\partial \varepsilon_{zz}}{\partial x_1} = \frac{\partial}{\partial z}\left(\frac{\partial u_z}{\partial x_1}\right)$$

$$\frac{\partial \varepsilon_{rz}}{\partial x_1} = \frac{1}{2}\left\{\frac{\partial}{\partial z}\left(\frac{\partial u_r}{\partial x_1}\right) + \frac{\partial}{\partial r}\left(\frac{\partial u_z}{\partial x_1}\right)\right\}, \tag{4}$$

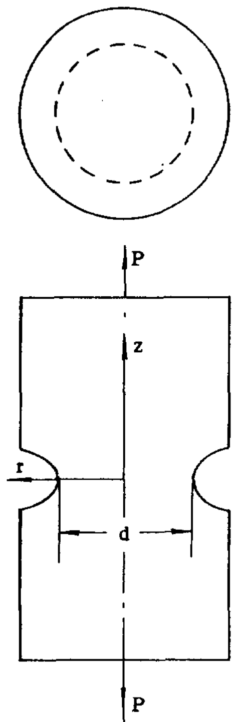

Fig. 2. A circular cylinder with a circular notch subjected to axial tension.

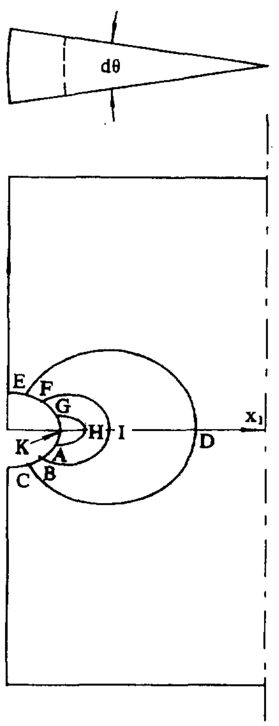

Fig. 3. Paths of integration of conservative integral for axially symmetric problems.

where x_1 is the abscissa of a new coordinate system, shown in Fig. 3. Furthermore, if σ_{rr}, $\sigma_{\theta\theta}$, σ_{zz} and σ_{rz} are the real stress components, then they will be statically admissible ones, i.e. eqn (2) can be satisfied. If T_r and T_z are used to denote the components of surface traction as shown in Fig. 3, then

$$T_r = \sigma_{rr} l_r + \sigma_{rz} l_z$$
$$T_z = \sigma_{rz} l_r + \sigma_{zz} l_z, \tag{5}$$

where l_r and l_z are direction cosines of the outward normal of the closed contour line Γ^* with respect to the x_1 and z axes, respectively. Under the above conditions, the following equation of virtual work will be valid:

$$\int_{A_\Gamma} \left(\sigma_{rr} \frac{\partial \varepsilon_{rr}}{\partial x_1} + \sigma_{\theta\theta} \frac{\partial \varepsilon_{\theta\theta}}{\partial x_1} + \sigma_{zz} \frac{\partial \varepsilon_{zz}}{\partial x_1} + \sigma_{rz} \frac{\partial 2\varepsilon_{rz}}{\partial x_1} \right) r \, \mathrm{d}A$$
$$= \int_{\Gamma^*} \left(T_r \frac{\partial u_r}{\partial x_1} + T_z \frac{\partial u_z}{\partial x_1} \right) r \, \mathrm{d}S + \int_{A_\Gamma} \left(F_r \frac{\partial u_r}{\partial x_1} + F_z \frac{\partial u_z}{\partial x_1} \right) r \, \mathrm{d}A. \tag{6}$$

Substituting eqn (3) into eqn (6) and applying the chain rule of differentiation and the transformation formula from area integral to contour integral, we have

$$\int_{\Gamma^*} W l_1 r \, \mathrm{d}S - \int_{A_\Gamma} W \frac{\partial r}{\partial x_1} \mathrm{d}A = \int_{\Gamma^*} \left(T_r \frac{\partial u_r}{\partial x_1} + T_z \frac{\partial u_z}{\partial x_1} \right) r \, \mathrm{d}S + \int_{A_\Gamma} \left(F_r \frac{\partial u_r}{\partial x_1} + F_z \frac{\partial u_z}{\partial x_1} \right) r \, \mathrm{d}A, \tag{7}$$

where l_1 is used to denote l_r.

It is known that $\Gamma^* = \Gamma - (-\Gamma_p)$, so we can define the conservative integral J from eqn (7) as follows:

$$J = \int_{-\Gamma_p} W l_1 r \, \mathrm{d}S - \int_{-\Gamma_p} \left(T_r \frac{\partial u_r}{\partial x_1} + T_z \frac{\partial u_z}{\partial x_1} \right) r \, \mathrm{d}S \tag{8}$$

$$J = \int_{\Gamma} W l_1 r \, \mathrm{d}S - \int_{A_\Gamma} W \frac{\partial r}{\partial x_1} \mathrm{d}A - \int_{\Gamma} \left(T_r \frac{\partial u_r}{\partial x_1} + T_z \frac{\partial u_z}{\partial x_1} \right) r \, \mathrm{d}S$$
$$+ \int_{A_\Gamma} \left(F_r \frac{\partial u_r}{\partial x_1} + F_z \frac{\partial u_z}{\partial x_1} \right) r \, \mathrm{d}A. \tag{9}$$

It is evident that the conservative integral given by eqn (9) is always equal to the conservative integral given by eqn (8). Furthermore, the path of integration Γ of eqn (9) is arbitrary and the path of integration Γ_p of eqn (8) is fixed. Therefore, the conservative integral J given by eqn (9) is path-independent.

For a bar with a circumferential notch free from surface traction and without body force, eqns (8) and (9) will become

$$J = \int_{-\Gamma_p} W l_1 r \, \mathrm{d}S \tag{10}$$

$$J = \int_{\Gamma} W l_1 r \, \mathrm{d}S - \int_{A_\Gamma} W \frac{\partial r}{\partial x_1} \mathrm{d}A - \int_{\Gamma} \left(T_r \frac{\partial u_r}{\partial x_1} + T_z \frac{\partial u_z}{\partial x_1} \right) r \, \mathrm{d}S, \tag{11}$$

respectively.

4. STRESS AND STRAIN CONCENTRATION

Let J^*, W^*, T_i^* and u_i^* denote J, W, T_i and u_i under linear elastic case recpectively. Then, from eqns (10) and (11), we have

$$J^* = \int_{-\Gamma_p} W^* l_1 r \, dS \tag{12}$$

$$J^* = \int_{\Gamma} W^* l_1 r \, dS - \int_{A_\Gamma} W^* \frac{\partial r}{\partial x_1} dA - \int_{\Gamma} \left(T_r^* \frac{\partial u_r^*}{\partial x_1} + T_z^* \frac{\partial u_z^*}{\partial x_1} \right) r \, dS. \tag{13}$$

From the statements in the above section, it can be shown that

$$W = W^*, \quad T_i = T_i^*, \quad u_i = u_i^* \quad \text{within } A_{\text{eu}} \text{ and along } \Gamma. \tag{14}$$

Then, from eqns (10)–(13), we have

$$\int_{-\Gamma_p} (W - W^*) l_1 r \, dS + \int_{A_p + A_{ei}} (W - W^*) \frac{\partial r}{\partial x_1} dA = 0. \tag{15}$$

Furthermore, a new function $g(r, z)$ is introduced as follows:

$$W(r, z) = g(r, z) W^*(r, z). \tag{16}$$

Let (r_k, z_k) denote the coordinates of point of stress and strain concentration K and carry out the expansion of $g(r, z)$ at point K. Then, we have

$$g(r, z) = \sum_{n=0}^{N} \frac{1}{n!} \left[(r - r_k) \frac{\partial}{\partial r} + (z - z_k) \frac{\partial}{\partial z} \right]^n g|_k. \tag{17}$$

For the case of small scale plasticity, in the domain of $A_p + A_{ei}$, taking the approximation of null order in eqn (17), we have

$$g(r, z) = g(r_k, z_k) = f, \quad W(r, z) = f W^*(r, z). \tag{18}$$

Substituting eqn (18) into eqn (15), it can be confirmed that

$$f = 1, \quad W(r, z) = W^*(r, z). \tag{19}$$

According to the traction-free conditions along the notch surface, we have

$$\sigma_{rr} = \sigma_{rz} = 0 \quad \text{at point } K(r_k, z_k) \tag{20}$$

and by means of the theorem of proportional loading in deformation theory of plasticity, the following equality is valid:

$$\frac{\sigma_{\theta\theta}}{\sigma_{zz}} = \frac{\sigma_{\theta\theta}^*}{\sigma_{zz}^*} = \alpha(r_k, z_k), \tag{21}$$

where $\sigma_{\theta\theta}^*$ and σ_{zz}^* are stress components at K in the linear elastic case.

The constitutive equations based upon deformation theory at point K will be

$$e_{zz} = \frac{3\varepsilon_e}{2\sigma_e} S_{zz} \quad e_{\theta\theta} = \frac{3\varepsilon_e}{2\sigma_e} S_{\theta\theta} \quad e_{rr} = \frac{3\varepsilon_e}{2\sigma_e} S_{rr} \quad \varepsilon_m = \frac{1}{3K}\sigma_m, \tag{22}$$

where

$$\begin{aligned} \varepsilon_m &= \tfrac{1}{3}(\varepsilon_{zz}+\varepsilon_{rr}+\varepsilon_{\theta\theta}) \\ \sigma_m &= \tfrac{1}{3}(\sigma_{zz}+\sigma_{\theta\theta}) \\ K &= \frac{E}{3(1-2\mu)} \end{aligned} \tag{23}$$

$$\begin{aligned} e_{zz} &= \varepsilon_{zz}-\varepsilon_m, \quad e_{rr} = \varepsilon_{rr}-\varepsilon_m, \quad e_{\theta\theta} = \varepsilon_{\theta\theta}-\varepsilon_m \\ S_{zz} &= \sigma_{zz}-\sigma_m, \quad S_{rr} = \sigma_{rr}-\sigma_m, \quad S_{\theta\theta} = \sigma_{\theta\theta}-\sigma_m \end{aligned} \tag{24}$$

$$\sigma_e = \sqrt{\tfrac{3}{2}(S_{zz}^2+S_{rr}^2+S_{\theta\theta}^2)} \quad \varepsilon_e = \sqrt{\tfrac{2}{3}(e_{zz}^2+e_{rr}^2+e_{\theta\theta}^2)}. \tag{25}$$

For a general hardening material, we have

$$\varepsilon_e = \frac{\sigma_e}{E}\left\{1+\sum_{n=1}^{N} b_n\sigma_e^{2n}\right\}, \tag{26}$$

and the strain energy density in the elasto-plastic range of deformation at stress concentration point K can be expressed as follows:

$$W_K = \int_0^M \sigma_e\,\mathrm{d}\varepsilon_e + \frac{\sigma_m^2}{2K} = \frac{1}{2E}\left\{\sigma_e^2+\sum_{N=1}^{N}\frac{2n+1}{n+1}b_n\sigma_e^{2(n+1)}\right\}+\frac{3(1-2\mu)}{2E}\sigma_m^2. \tag{27}$$

For the linear elastic case, it is known that at point K

$$\begin{aligned} \varepsilon_{zz}^* &= \frac{1}{E}(\sigma_{zz}^*-\mu\sigma_{\theta\theta}^*) \\ \varepsilon_{\theta\theta}^* &= \frac{1}{E}(\sigma_{\theta\theta}^*-\mu\sigma_{zz}^*) \end{aligned} \tag{28}$$

$$W_K^* = \tfrac{1}{2}(\sigma_{zz}^*\varepsilon_{zz}^*+\sigma_{\theta\theta}^*\varepsilon_{\theta\theta}^*) = \frac{1}{2E}(\sigma_{zz}^{*2}-2\mu\sigma_{zz}^*\sigma_{\theta\theta}^*+\sigma_{\theta\theta}^{*2}). \tag{29}$$

Substituting eqn (21) into eqns (23), (24), (27), (29) and utilizing eqn (19), the following will be valid:

$$\beta\sigma_{zz}^2+\sum_{n=1}^{N}\beta_n\sigma_{zz}^{2(n+1)} = \sigma_{zz}^{*2}, \tag{30}$$

where

$$\beta = \frac{3(1-\alpha+\alpha^2)+(1-2\mu)(1+\alpha)^2}{3(1-2\mu\alpha+\alpha^2)}$$

$$\beta_n = \frac{2n+1}{n+1} b_n \frac{(1-\alpha+\alpha^2)^{n+1}}{1-2\mu\alpha+\alpha^2}. \tag{31}$$

We define the following concentration factors:

$$K_z^* = \frac{\sigma_{zz}^*}{\sigma_n}, \quad K_\theta^* = \frac{\sigma_{\theta\theta}^*}{\sigma_n}, \quad K_T = \frac{\sigma_e^*}{\sigma_n}$$

$$K_z = \frac{\sigma_{zz}}{\sigma_n}, \quad K_\theta = \frac{\sigma_{\theta\theta}}{\sigma_n}, \quad K_z' = \frac{\varepsilon_{zz}}{\varepsilon_n}$$

$$K_\sigma = \frac{\sigma_e}{\sigma_n}, \quad K_\varepsilon = \frac{\varepsilon_e}{\varepsilon_n}, \tag{32}$$

where σ_n and $\varepsilon_n = \sigma_n/E$ are nominal stress and strain, respectively.

From eqns (21), (23), (24) and (25), we have

$$\sigma_e = \sigma_{zz}\sqrt{1-\alpha+\alpha^2}$$

$$\sigma_e^* = \sigma_{zz}^*\sqrt{1-\alpha+\alpha^2}. \tag{33}$$

According to eqns (21) and (32), it can be obtained that

$$K_z = \frac{\sigma_{zz}}{\sigma_{zz}^*} K_z^*, \quad K_\theta = \alpha K_z. \tag{34}$$

Furthermore, considering eqn (33), we have

$$K_\sigma = K_z\sqrt{1-\alpha+\alpha^2}$$

$$K_T = K_z^*\sqrt{1-\alpha+\alpha^2}. \tag{35}$$

Finally, from eqns (21)–(26), it can be shown that

$$K_z' = K_z\left\{\frac{2-\alpha}{2}\left(1+\sum_{n=1}^{N} b_n\sigma_e^{2n}\right)+\frac{1-2\mu}{3}(1+\alpha)\right\} \tag{36}$$

$$K_\varepsilon = K_\sigma\left(1+\sum_{n=1}^{N} b_n\sigma_e^{2n}\right). \tag{37}$$

The solution procedure can be arranged as follows:

(i) determine β and β_n by means of eqn (31) with given material constants b_n and field constant α;
(ii) calculate a sequence of magnitude of σ_{zz}^* corresponding to a sequence of given magnitude of σ_{zz} by means of eqn (30);
(iii) draw a σ_{zz}–σ_{zz}^* curve with the above data;
(iv) compute σ_e, K_z, K_θ, K_σ, K_T, K_z' and K_ε by means of eqns (33)–(37).

The most important thing in this method of solution is the establishment of the relation between σ_{zz} and σ_{zz} at the point K of stress and strain concentrations.

It must be emphasized that by means of this solution procedure, the difficulties in the solution of nonlinear algebraic equation (30) can be avoided.

5. COMPUTATIONAL EXAMPLES

If the generating line of a notch is a hyperbolic one, then we have (see e.g. Neuber (1947))]

$$\alpha = \frac{B}{A}, \quad K'_z = \frac{A}{N}, \tag{38}$$

where

$$A = \frac{d}{2\rho}\sqrt{\frac{d}{2\rho}+1}+(\tfrac{1}{2}+\mu)\frac{d}{2\rho}+1+(1+\mu)\left(\sqrt{\frac{d}{2\rho}+1}\right) \tag{39}$$

$$B = \frac{d}{2\rho}\left(\mu\sqrt{\frac{d}{2\rho}+1}+\frac{1}{2}\right) \tag{40}$$

$$N = \frac{d}{2\rho}+2\mu\sqrt{\frac{d}{2\rho}+1}+2. \tag{41}$$

In the above equations, d is the minimum diameter of the circular bar, ρ is the radius of curvature at the notch bottom, as shown in Fig. 1, and μ is Poisson's ratio.

For conventional hardening materials, it is sufficient to take three terms in eqn (26), then we have

$$\varepsilon_e = \frac{\sigma_e}{E}(1+b_1\sigma_e^2+b_2\sigma_e^4), \tag{42}$$

where b_1 and b_2 are shown in Table 1.

By means of eqns (38)–(41), we can obtain the α–$d/2\rho$ curves for various values of μ, as shown in Fig. 4.

Figure 5 shows relations between σ_{zz} and σ_{zz}^* for different values of μ and $d/2\rho$. It is noted that the curve for $d/2\rho = 10$ can be used for the case for $d/2\rho > 10$, because α will remain constant when $d/2\rho > 10$, as shown in Fig. 4, and so β and β_n will be unchanged when $d/2\rho > 10$ from eqn (31).

Figure 6 shows plots of K_z–σ_{zz}/σ_u and K_z–σ_{zz}/σ_u for different values of μ and $d/2\rho$. From this figure, it can be seen that K_z is not sensitive to μ but K'_z is very sensitive to it. σ_u is used to denote the ultimate strength of the material.

Table 1. Data of material constants

Materials	30CrMnSiNi2A	LC9 (Al alloy)
b_1[GPa]$^{-2}$	−0.251417	−1.835680
b_2[GPa]$^{-4}$	0.338404	20.009000
σ_u[GPa]	1.6520	0.5230

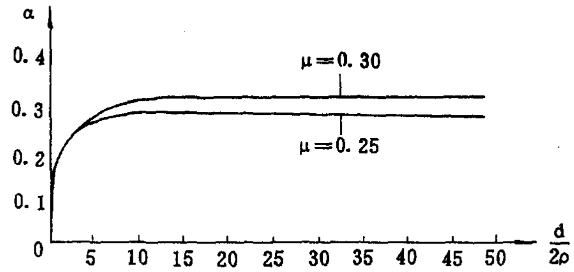

Fig. 4. α–$d/2\rho$ curves.

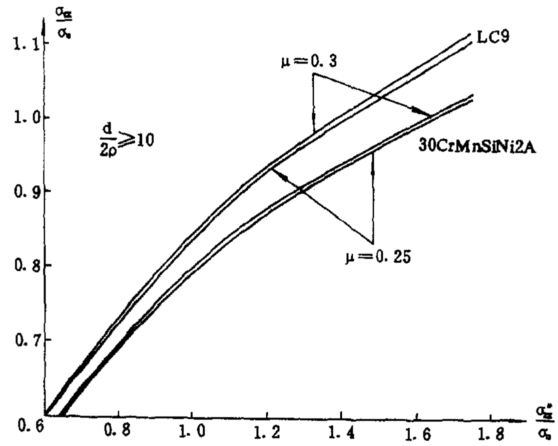

Fig. 5. σ_{zz}/σ_u–σ^*_{zz}/σ_u curves.

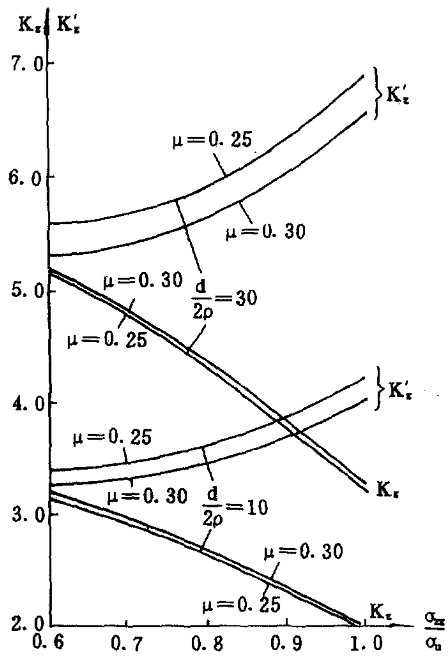

Fig. 6. K_ε–σ_{zz}/σ_u curves and K'_ε–σ_{zz}/σ_u curves.

To verify the availability of closed form solutions by the method of conservative integrals, comparisons between the results given by this method and following four formulas are carried out.

(i) ASME Code formula [see e.g. Seeger *et al.* (1977)]:

$$\frac{K_\varepsilon}{K_\sigma}=\frac{K_T}{K_\sigma};\tag{43}$$

(ii) Dietmann–Saal formula [see e.g. Seeger *et al.* (1977)]:

$$\frac{K_\varepsilon}{K_\sigma}=\left\{\frac{K_p-1}{K_p-K_T/K_\sigma}\right\}^{K_\sigma-1};\tag{44}$$

(iii) Stowell–Hardrath–Ohman formula [see e.g. Seeger *et al.* (1977)]:

$$\frac{K_\varepsilon}{K_\sigma}=\left\{\frac{K_P-1}{K_P-K_T/K_\sigma}\right\}\frac{K_T}{K_\sigma}\tag{45}$$

(iv) Neuber's formula [see e.g. Neuber (1961)]:

$$\frac{K_\varepsilon}{K_\sigma}=\left\{\frac{K_T}{K_\sigma}\right\}^2.\tag{46}$$

In eqns (43)–(45),

$$K_P=\frac{S_P}{S_S},\tag{47}$$

where S_P is the nominal limiting stress, i.e. the limit load for an ideal plastic material divided by the minimum cross-section area, and S_S is the nominal yield stress, i.e. the yield stress divided by the elastic stress concentration factor.

The results of comparisons are shown in Figs 7–10. In these figures,

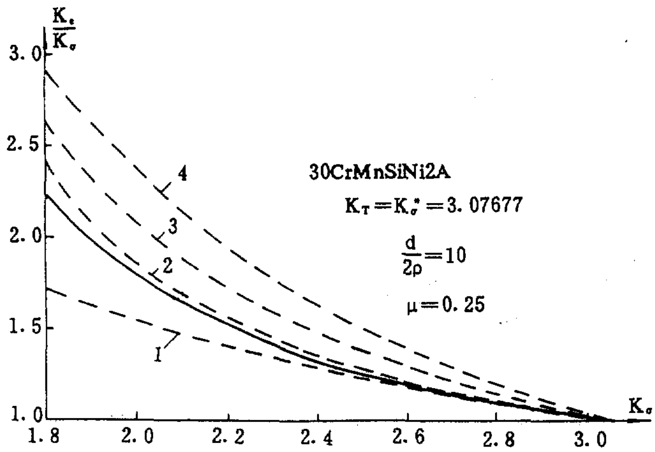

Fig. 7. K_ε/K_σ–K_σ curves of 30CrMnSiNi2A ($d/2\rho=10$).

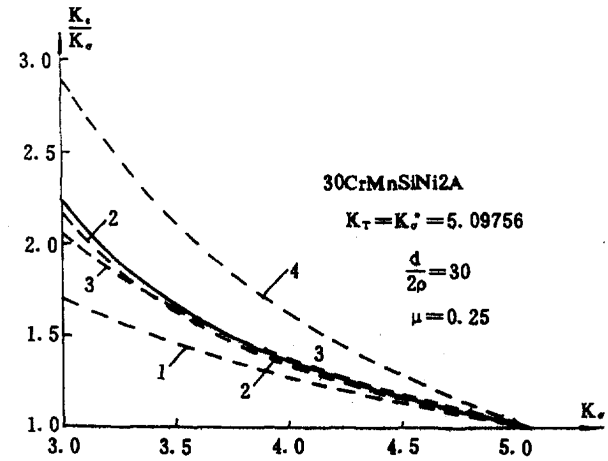

Fig. 8. K_ε/K_σ–K_σ curves for 30CrMnSiNi2A ($d/2\rho = 30$).

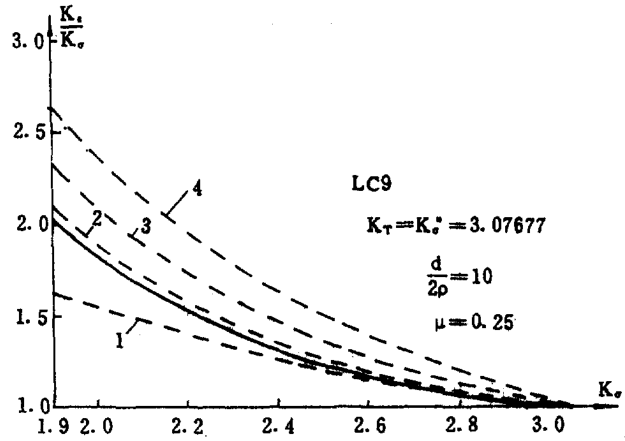

Fig. 9. K_ε/K_σ–K_σ curves for LC9 ($d/2\rho = 10$).

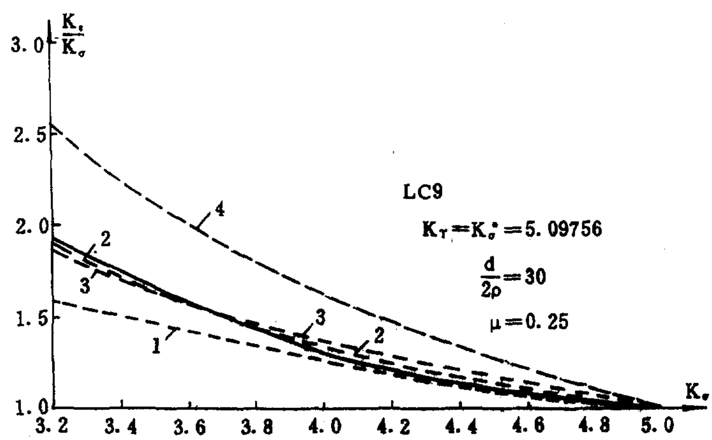

Fig. 10. K_ε/K_σ–K_σ curves for 30CrMnSiNi2A ($d/2\rho = 30$).

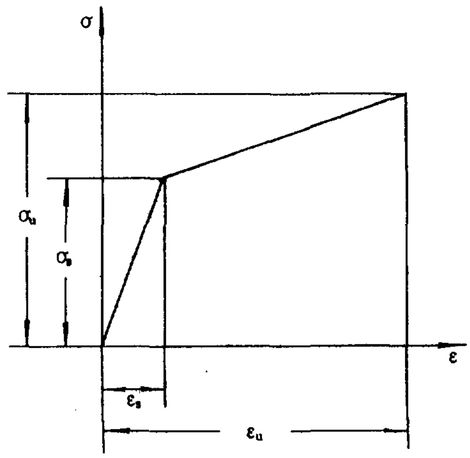

Fig. 11. Stress–strain curve of linear hardening material.

$$K_T = 3.07677 \leftrightarrow \frac{d}{2\rho} = 10, \quad \mu = \tfrac{1}{4};$$

$$K_T = 5.09756 \leftrightarrow \frac{d}{2\rho} = 30, \quad \mu = \tfrac{1}{4};$$

solid lines are used to denote the results given by this method and dotted lines are applied to express the results obtained by other methods.

From these figures, it can be seen that the results given by this method fit those obtained by eqns (44) and (45) very well and agree with the average values given by the other four formulas satisfactorily.

It should be emphasized that among the four other formulas, the first three are empirical and the latter was derived under the condition of anti-plane shear for power hardening materials. So, the method provided in this paper is significant not only in the sense of engineering application but also in the sense of theoretical investigation.

6. EVALUATION OF ULTIMATE LOAD

Ultimate load is defined as the load P_u when effective stress σ_e at the point of stress concentration K arrives at the ultimate strength σ_u. For convenience of calculation, the materials are assumed to be linear hardening, as shown in Fig. 11. Then, we have

$$\begin{aligned} \varepsilon_e &= \frac{\sigma_e}{E}, \quad \sigma_e \leqslant \sigma_s \\ \varepsilon_e &= \frac{1}{E}\{\sigma_s + m(\sigma_e - \sigma_s)\}, \quad \sigma_s < \sigma_e \leqslant \sigma_u, \end{aligned} \tag{48}$$

where

$$m = E\frac{\varepsilon_u - \varepsilon_s}{\sigma_u - \sigma_s}. \tag{49}$$

In the above formula, ε_u and ε_s are strains corresponding to σ_u and σ_s, respectively.

According to eqns (27) and (29), the strain energy density at point K will be

$$W_K = \frac{1}{2E}\left\{(1-m)\sigma_s^2 + \left[m + \frac{(1-2\mu)(1+\alpha)^2}{3(1-\alpha+\alpha^2)}\right]\sigma_u^2\right\} \tag{50}$$

$$W_K^* = \frac{K_z^* \sigma_n^*}{2E}(1-2\mu\alpha+\alpha^2), \tag{51}$$

where σ_n is the nominal ultimate strength

$$\sigma_n^* = \frac{4P_u}{\pi d^2}. \tag{52}$$

Substituting eqns (50) and (51) into eqn (19), we have

$$P_u = \frac{\pi d^2}{4K_z^*}\sqrt{\frac{(1-m)\sigma_s^2 + \left[\frac{(1-2\mu)(1+\alpha)^2}{3(1-\alpha+\alpha^2)}\right]\sigma_u^2}{1-2\mu\alpha+\alpha^2}}. \tag{53}$$

REFERENCES

Neuber, H. (1961). *Trans. Ser. E, J. Appl. Mech.* **28,** 544–550.
Seeger, T., Beste, A. and Amstutz, H. (1977). *Fracture* **2B,** 943–952.
НеиЬер, Г. (1947). *Stress concentration* ОТИЗ ГОСТЕХИЗДАТ, 99–104.